

On topological actions of finite, non-standard groups on spheres

Bruno P. Zimmermann

Università degli Studi di Trieste
Dipartimento di Matematica e Geoscienze
34127 Trieste, Italy

Abstract. *The standard actions of finite groups on spheres S^d are linear actions, i.e. by finite subgroups of the orthogonal groups $O(d+1)$. We prove that, in each dimension $d > 5$, there is a finite group G which admits a faithful, topological action on a sphere S^d but is not isomorphic to a subgroup of $O(d+1)$. The situation remains open for smooth actions.*

1. Introduction

We are interested in faithful actions by homeomorphisms of finite groups on spheres S^d . The standard actions on spheres are the linear or orthogonal actions by the finite subgroups of the orthogonal group $O(d+1)$. There is a rich literature on smooth, non-linear actions of finite groups on spheres (see the surveys [D], [Z1]), but not much is known on the class of finite groups which can occur for smooth or topological actions; in particular, not a single example of a finite group seems to be known which admits a faithful, smooth action on a sphere S^d but does not admit a faithful, linear action on S^d (i.e., is not isomorphic to a subgroup of $O(d+1)$). Concerning topological actions, our main result is the following.

Theorem. *For each dimension $d > 5$, there is a finite group G which admits a faithful, topological action on the sphere S^d but is not isomorphic to a subgroup of $O(d+1)$.*

In fact, relying on an observation in [CKS], our methods would imply that in each dimension $d \geq 5$ there are infinitely many such groups, see the remark at the end of section 2. We note that the actions we construct are equivalent to simplicial actions but they are not locally linear and hence not equivalent to smooth actions.

The Theorem remains open in dimension 3. In dimension 3, as a consequence of the geometrization of finite group actions on 3-manifolds after Thurston and Perelman (cf. [BLP],[DL]), every finite group which admits a faithful, smooth action on S^3 is isomorphic (and even conjugate) to a subgroup of $O(4)$ (but this does not remain true for smooth actions on homology 3-spheres, see [Z2] and [Z3, section 5] for a discussion). The major problem for topological actions on the 3-sphere is then the possible presence

of wildly embedded fixed point sets; for triangulable or PL actions, this phenomenon does not occur, the actions are then locally linear, conjugate to smooth actions and, by the geometrization, also to linear ones. On the other hand, almost nothing seems to be known on the possible finite groups which can act on S^3 with wild fixed point sets.

Concerning dimension 4, it is proved in [CKS] (completing results in [MeZ1,2]) that every finite group which admits a smooth or locally linear, orientation-preserving, faithful action on S^4 or on a homology 4-sphere is isomorphic to a subgroup of $\mathrm{SO}(5)$, but that this does not remain true for topological actions on S^4 (the orientation-reversing analogue is still open in the smooth case, and again not true in the topological case).

Returning to arbitrary dimensions, we have the following:

Question. Is there a finite group G which admits a faithful, smooth action on a sphere S^d but is not isomorphic to a subgroup of $\mathrm{O}(d+1)$? Does the Theorem remain true for faithful, smooth actions?

We note that, for certain classes of finite groups G , the minimal dimension of a faithful, smooth action of G on a homology sphere coincides with the minimal dimension of a faithful, linear action on a sphere, e.g. for the linear fractional groups $\mathrm{PSL}_2(p)$ ([GZ, Theorem 3]), for finite p -groups ([DoH]), for some classes of alternating groups and some other finite simple groups (and in some cases also for purely topological actions, using Smith fixed point theory and the Borel formula which hold in a purely topological setting).

The proof of the Theorem is based on the existence, due to Milgram [Mg], of a finite group Q (Milnor group) which admits a faithful, smooth action on a homology 3-sphere M^3 but is not isomorphic to a subgroup of the orthogonal group $\mathrm{O}(4)$, and also on the double suspension theorem stating that the double suspension or join $M^3 * S^0 * S^0 \cong M^3 * S^1$ (see e.g. [Mu]) of a homology 3-sphere M^3 is homeomorphic to S^5 (see [Ca]); so this allows to shift finite actions on homology 3-spheres to topological actions on spheres in higher dimensions.

2. Proof of the Theorem

By strong results of Milgram [Mg], there exists a finite group Q which admits a smooth, faithful action on a homology 3-sphere M^3 but is not isomorphic to a subgroup of the orthogonal group $\mathrm{O}(4)$ (so Q does not admit a faithful, linear action on S^3). The group Q is a Milnor group $Q(8a, b, c)$ ([Mn]), for relatively coprime odd positive integers a, b and c with $a \geq 3$ and $b > c \geq 1$. Such a Milnor group $Q(8a, b, c)$ is an extension of $\mathbb{Z}_{abc} \cong \mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c$ by the quaternion group $Q(8) = \{\pm 1, \pm i, \pm j, \pm k\}$ of order 8, where i, j and k act trivially on $\mathbb{Z}_a, \mathbb{Z}_b$ and \mathbb{Z}_c , respectively, and in a dihedral way on the other two. By [Mn], $Q(8a, b, c)$ is not isomorphic to a subgroup of $\mathrm{O}(4)$ (see also [Z3, section 3] for a discussion and other references).

By the double suspension theorem (see [Ca]), for $m \geq 1$ the join

$$M^3 * S^m \cong M^3 * S^1 * S^{m-2} \cong S^5 * S^{m-2} \cong S^{m+4}$$

is homeomorphic to S^{m+4} .

The alternating group \mathbb{A}_n has a linear, faithful action on \mathbb{R}^n by permutation of coordinates, and also on \mathbb{R}^{n-1} (in coordinates, on the subspace of \mathbb{R}^n given by $x_1 + \dots + x_n = 0$), and hence on S^{n-2} . The group $G = Q \times \mathbb{A}_n$ has then a faithful action on the join $M^3 * S^{n-2} \cong S^{n+2}$, by joining the actions of Q on M^3 and of \mathbb{A}_n on S^{n-2} (with Q acting trivially on S^{n-2} and \mathbb{A}_n trivially on M^3).

So G admits a faithful, topological action on S^{n+2} (which is not locally linear). We will show that G does not admit a faithful, linear action on S^{n+2} . We fix a linear action of G on S^{n+2} and suppose, by contradiction, that the action is faithful.

The linear action of G on $S^{n+2} \subset \mathbb{R}^{n+3}$ defines a $(n+3)$ -dimensional real representation of G . The induced linear representation of \mathbb{A}_n on \mathbb{R}^{n+3} splits as a direct sum of irreducible representations. Suppose first that $n \geq 7$; then the only irreducible representations of \mathbb{A}_n in dimensions smaller than $n+3$ are the trivial representation and the standard representation in dimension $n-1$, so there is an orthogonal decomposition $\mathbb{R}^{n+3} = \mathbb{R}^4 \oplus \mathbb{R}^{n-1}$ where \mathbb{A}_n acts trivially on the first summand and by the standard representation on the second one. The group Q preserves this decomposition and commutes elementwise with \mathbb{A}_n . Complexifying, we have a G -invariant decomposition $\mathbb{C}^{n+3} = \mathbb{C}^4 \oplus \mathbb{C}^{n-1}$; then, by Schur's Lemma, Q acts by homotheties on the second summand \mathbb{C}^{n-1} , i.e. by scalar multiples of the identity (see [S] or [FH] for the representation theory of finite groups), and hence the action of Q on \mathbb{C}^{n-1} factors through the action of an abelian group. The abelianization of Q is the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$, generated by the images of i and j ; in particular, the central involution -1 and the cyclic subgroup \mathbb{Z}_{abc} of Q act trivially on \mathbb{C}^{n-1} , and hence also on \mathbb{R}^{n-1} .

Now we consider the action of Q on the first summand \mathbb{R}^4 . Since the action of Q on $\mathbb{R}^{n+3} = \mathbb{R}^4 \oplus \mathbb{R}^{n-1}$ is faithful, the central involution -1 and the cyclic subgroup \mathbb{Z}_{abc} of Q act faithfully on the first summand \mathbb{R}^4 ; then also i, j, k and the subgroup $Q(8)$ of Q (the quaternion group of order 8) act faithfully on \mathbb{R}^4 , and hence the whole group Q . Since Q is not isomorphic to a subgroup of $O(4)$ (see [Mn] and [Z3, section 3]), this is a contradiction, so the action of G on S^{n+2} cannot be faithful.

We have proved the Theorem for $n \geq 7$, or $d \geq 9$. The cases $d = 8, 7$ and 6 are similar, considering the alternating and symmetric groups $\mathbb{A}_6, \mathbb{S}_5$ and \mathbb{A}_5 and their irreducible representations (see [Co] for their character tables).

This concludes the proof of the Theorem.

Remarks. i) We discuss the case $d = 5$. The Milnor group Q considered before admits a faithful action on $M^3 * S^1 \cong S^5$ (with the trivial action on S^1). Now the authors of

[CKS] remark (in a note in section 2) that a Milnor group $Q(8a, b, c)$ is not isomorphic to a subgroup of $O(m)$, for $m \leq 7$ (indicating an idea of a proof), so this would imply that Q does not admit a faithful, linear action on a sphere of dimension less than seven. The same holds then for the infinitely many groups $Q \times \mathbb{Z}_k$; on the other hand, these groups admit a faithful, topological action on $M^3 * S^1 \cong S^5$, by letting \mathbb{Z}_k act faithfully by rotations on the second factor.

ii) Elaborating on this, there is a faithful, topological action of a group $G = Q \times \mathbb{Z}_k \times \mathbb{A}_n$ on $M^3 * S^1 * S^{n-2} \cong S^5 * S^{n-2} \cong S^{n+4}$. Now, if Q is not a subgroup of $O(6)$, exactly as in the proof of the Theorem one shows that G does not admit a faithful, linear action on S^{n+4} , so in each dimension $d \geq 5$ there are infinitely many groups as in the Theorem (considering the groups $Q \times \mathbb{Z}_k$ in dimensions $d < 7$).

iii) In the proof of the theorem, the action of Q on $M^3 * S^{n-2} \cong S^{n+2}$, with fixed point set S^{n+2} , is not locally linear (otherwise Q , which is not a subgroup of the orthogonal group $O(4)$, would act faithfully and orthogonally on a 3-sphere orthogonal to the fixed point set). Choose a surjection $Q \rightarrow \mathbb{Z}_2$ and let a generator of \mathbb{Z}_2 act by minus identity on S^{n-2} ; this defines an action of Q on S^{n-2} without a global fixed point. The kernel of the has fixed point set S^{n-2} and is isomorphic to a subgroup of $O(4)$: is the combined action of Q on $M^3 * S^{n-2}$ locally linear now (e.g., in the case $M^3 * S^1 \cong S^5$)?

On the other hand, also the action of \mathbb{A}_n on $M^3 * S^{n-2}$ is not locally linear: if an element of \mathbb{A}_n has 0-dimensional fixed point set S^0 in S^{n-2} then its fixed point set in $M^3 * S^{n-2}$ is the suspension or double cone $M^3 * S^0$ which is a homology manifold but not a manifold in the two cone points. So, in order to construct a locally linear action of a group $Q \times A$ on $M^3 * S^{n-2}$, one should avoid such low-dimensional fixed point sets for an action of some group A on S^{n-2} .

iv) The group Q admits a free action on $M^3 * M^3$ which is homeomorphis to S^7 (by the solution of the higher-dimensional Poincaré conjecture): is this action conjugate to a free linear action of Q on S^7 (noting that Q occurs as a fixed-point free subgroup of $SU(4)$ and hence $O(8)$)? Note that Q has a free action also on $S^{11} \cong S^7 * M^3 \cong M^3 * M^3 * M^3$ but no free linear action on S^{11} .

Finally, let M^3 be any homology 3-sphere with a free action of the cyclic group \mathbb{Z}_p . Letting \mathbb{Z}_p act by rotations on S^1 , it has a free action on $M^3 * S^1 \cong S^5$: is this action conjugate to a linear action?

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